Decomposable branching processes having a fixed extinction moment*

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April 21, 2015

Abstract

The asymptotic behavior, as $n \to \infty$ of the probability of the event that a decomposable critical branching process $\mathbf{Z}(m) = (Z_1(m), ..., Z_N(m))$, $m = 0, 1, 2, ...$, with N types of particles dies at moment n is investigated and conditional limit theorems are proved describing the distribution of the number of particles in the process $\mathbf{Z}(\cdot)$ at moment $m < n$, given that the extinction moment of the process is n.

These limit theorems may be considered as the statements describing the distribution of the number of vertices in the layers of certain classes of simply generated random trees having a fixed hight.

AMS Subject Classification: 60J80, 60F99, 92D25

Key words: decomposable branching processes, criticality, extinction, limit theorems, random trees

1 Introduction

We consider a Galton-Watson branching process with N types of particles labelled $1, 2, ..., N$ and denote by

$$\mathbf{Z}(n) = (Z_1(n), ..., Z_N(n))$$

the population vector at time $n \in \mathbb{Z}_+ = \{0, 1, ...\}, \mathbf{Z}(0) = (1, 0, ..., 0)$. Denote by T_N the extinction moment of the process. The aim of the present paper is to investigate the asymptotic behavior, as $n \to \infty$ of the probability of the event $\{T_N = n\}$ and the distribution of the random vector $\mathbf{Z}(m), 0 \le m < n$, given $T_N = n$ and assuming that $\mathbf{Z}(\cdot)$ is a decomposable critical branching process.

Properties of the single-type critical Galton-Watson process given its extinction moment have been investigated by a number of authors (see, for instance,

*This work is supported by the RSF under a grant 14-50-00005.

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[5], [6], [10]). Asymptotic properties of the survival probability for multitype indecomposable critical Markov processes as well as the properties of these processes given their survival up to a distant moment were analysed in [4], [8] and [14].

The decomposable branching processes are less investigated. We mention papers [1], [2], [9], [11], [12], [13], [16], [17], [18], [19] in this connection where the asymptotic representations for the probability of the event $\{T_N > n\}$ are found under various restrictions and the Yaglom-type limit theorems for the distribution of the number of particles are proved for the multi-type decomposable critical Markov processes (and their reduced analogues) under the condition $T_N > n$. However, the study of the asymptotic properties of the probability $\mathbf{P}(T_N = n)$ for the decomposable critical Markov branching processes and investigation of the conditional distributions of the number of particles in these processes given $T_N = n$, have not been considered up to now. The present paper deals with such circle of questions.

Namely, we consider a decomposable Galton-Watson branching process with N types of particles in which a type i parent-particle may produce children of types $j \geq i$ only.

Introduce the probability generating functions for the distribution laws of the offspring sizes of particles

$$h^{(i,N)}(\mathbf{s}) = h^{(i,N)}(s_i, ..., s_N) = \mathbf{E}\left[s_i^{\eta_{i,i}} ... s_N^{\eta_{i,N}}\right], \ i = 1, 2, ..., N, \tag{1}$$

where the random variable $\eta_{i,j}$ is equal to the number of type j daughter particles of a type i particle.

Let $\mathbf{e}_i$ be an N-dimensional vector whose i-th component is equal to one while the remaining are zeros and $\mathbf{0} = (0, ..., 0)$ be an N–dimensional vector all whose components are zeros. The first moments of the components of $\mathbf{Z}(n)$ will be denoted as

$$m_{i,j}(n) = \mathbf{E}\left[Z_j(n) \left|\mathbf{Z}_0 = \mathbf{e}_i\right.\right]$$

with $m_{i,j} = m_{i,j}(1) = \mathbf{E}[\eta_{i,j}]$ being the average number of type j children produced by a particle of type i.

Since $m_{i,j} = 0$ if $i > j$, the mean matrix $\mathbf{M} = (m_{i,j})_{i,j=1}^N$ of our decomposable Galton-Watson branching process has the form

$$\mathbf{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & ... & ... & m_{1,N} \\ 0 & m_{2,2} & ... & ... & m_{2,N} \\ 0 & 0 & m_{3,3} & ... & ... \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & m_{N,N} \end{pmatrix}. \tag{2}$$

We say that **Hypothesis A** is valid if the decomposable branching process with N types of particles is strongly critical, i.e. (see [2])

$$m_{i,i} = \mathbf{E}[\eta_{i,i}] = 1, \quad i = 1, 2, ..., N \tag{3}$$

and, in addition,

$$m_{i,i+1} = \mathbf{E}\left[\eta_{i,i+1}\right] \in (0,\infty),\ \ i = 1,2,...,N-1, \tag{4}$$

$$\mathbf{E}\left[\eta_{i,j}\eta_{i,k}\right] < \infty,\ i = 1,...,N;\ \ k,j = i,i+1,...,N, \tag{5}$$

with

$$b_i = \frac{1}{2} Var\left[\eta_{i,i}\right] \in (0,\infty)\,, i = 1,2,...,N. \tag{6}$$

Thus, a particle of the process is able to produce the direct descendants of its own type, of the next in the order type, and (not necessarily, as direct descendants) of all the remaining in the order types, but not any preceding ones.

In the sequel we assume (if otherwise is not stated) that $\mathbf{Z}(0) = \mathbf{e}_1$, i.e. we suppose that the branching process in question is initiated at time $n = 0$ by a single particle of type 1.

The functions $\Phi_i = \Phi_i(\lambda_i, \lambda_{i+1}, ..., \lambda_N),\ i = 1,2,...,N-1$, being for $\lambda_j \geq 0, j = 1,2,...,N$ solutions of the equations

$$\sum_{k=i}^{N} (k-i+1)\,\lambda_k \frac{\partial \Phi_i}{\partial \lambda_k} = -b_i \Phi_i^2 + \Phi_i + \sum_{k=i}^{N} f_{k,i}\lambda_k,\ \ i = 1,2,...,N-1, \tag{7}$$

with the initial conditions

$$\Phi_i(\mathbf{0}) = 0,\ \ \frac{\partial \Phi_i(\mathbf{0})}{\partial \lambda_i} = 1$$

and, for $k > i$

$$\frac{\partial \Phi_i(\mathbf{0})}{\partial \lambda_k} = \frac{f_{k,i}}{k-i} = \frac{1}{(k-i)!} \prod_{j=i}^{k-1} m_{j,j+1}$$

are important in the statements of the theorems to follow. Existence and uniqueness of the solutions of the mentioned equations are established in [2]. Note that if $N = 2$, then

$$\Phi_2(\lambda_1,\lambda_2) = \sqrt{\frac{m_{1,2}\lambda_2}{b_1}} \frac{b_1\lambda_1 + \sqrt{b_1 m_{1,2}\lambda_2}\tanh\sqrt{b_1 m_{1,2}\lambda_2}}{b_1\lambda_1\tanh\sqrt{b_1 m_{1,2}\lambda_2} + \sqrt{b_1 m_{1,2}\lambda_2}}. \tag{8}$$

Let

$$c_{i,N} = \left(\frac{1}{b_N}\right)^{1/2^{N-i}} \prod_{j=i}^{N-1} \left(\frac{m_{j,j+1}}{b_j}\right)^{1/2^{j-i+1}}, \tag{9}$$

$$D_i = \left(b_i m_{i,i+1}\right)^{1/2^i} c_{1,i}, i = 1,2,...,N. \tag{10}$$

Denote

$$T_{ki} = \min\left\{n \geq 1 : Z_k(n) + Z_{k+1}(n) + ... + Z_i(n) = 0 | \mathbf{Z}(0) = \mathbf{e}_k\right\}$$

the extinction moment of the population consisting of the particles of types $k, k+1, ..., i$, given that the process was initiated at moment $n = 0$ by a single particle of type k. To simplify notation, set $T_i = T_{1i}$.

We fix $N \geq 2$ and use, when it is necessary, the notation

$$\gamma_0 = 0, \;\; \gamma_i = \gamma_i(N) = 2^{-(N-i)}, \;\; i = 1, 2, ..., N.$$

Besides, we write $a(n) \sim b(n)$ if $\lim_{n\to\infty} a(n)/b(n) = 1$ and $a(n) \ll b(n)$ if $\lim_{n\to\infty} a(n)/b(n) = 0$.

Asymptotic properties of the probability that a critical decomposable Galton-Watson branching process dies out at a fixed moment are described by the following theorem.

Theorem 1 *If Hypothesis A is valid, then*

$$\mathbf{P}\left(T_{iN} = n\right) \sim \frac{g_{i,N}}{n^{1+\gamma_i}}, \quad i = 1, 2, ..., N,$$

where

$$g_{i,N} = \gamma_i c_{i,N}.$$

We now formulate four more theorems in which, given $T_N = n$ the limiting distributions of the number of particles at moment m are found depending on the ratio between the parameters m and n.

Theorem 2 *If $n^{\gamma_1} \gg m \to \infty$, then*

$$\lim_{m\to\infty} \mathbf{E}\left[\exp\left\{-\sum_{l=1}^{N} \lambda_l \frac{Z_l(m)}{m^l}\right\} \Big| \, T_N = n\right] = \frac{\partial \Phi_1(\lambda_1, \lambda_2, ..., \lambda_N)}{\partial \lambda_1}.$$

We see that, given $\{T_N = n\}$ particles of all types present in the process at the initial stage of its evolution.

Theorem 3 *If $m \sim y n^{\gamma_i}$ for some $y > 0$ and $i \in \{1, 2, ..., N-1\}$, then, for any $s_k \in [0, 1]$, $k = 1, ..., i-1$, and $\lambda_l \geq 0, l = i, ..., N$*

$$\lim_{m\to\infty} \mathbf{E}\left[s_1^{Z_1(m)} \cdots s_{i-1}^{Z_{i-1}(m)} \exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\} \Big| \, T_N = n\right]$$
$$= D_{i-1} \frac{g_{i,N}}{g_{1,N}} \frac{\partial}{\partial \lambda_i} \left(\frac{\Phi_i(\lambda_i' y, \lambda_{i+1}' y^2, \lambda_{i+2} y^3, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^i}$$
$$+ D_{i-1} \frac{g_{i+1,N}}{g_{1,N}} \frac{\partial}{\partial \lambda_{i+1}} \left(\frac{\Phi_i(\lambda_i' y, \lambda_{i+1}' y^2, \lambda_{i+2} y^3, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^i},$$

where $\lambda_i' = \lambda_i + c_{i,N}, \lambda_{i+1}' = \lambda_{i+1} + c_{i+1,N}$.

Observe that if $i > 1$, then, under the conditions of Theorem 3 there are no particles of the types $1, 2, ..., i-1$ in the limit.

Let $a_{i,i} = 1$ and for $i < j$

$$a_{i,j} = \frac{1}{(j-i)!} \prod_{k=i}^{j-1} m_{k,k+1}. \tag{11}$$

Theorem 4 *If $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ for some $i \in \{1,2,...,N-1\}$, then, for any $s_k, k = 1,2,...,i$ and $\lambda_l \geq 0, l = i+1,...,N$*

$$\lim_{m\to\infty} \mathbf{E}\left[s_1^{Z_1(m)} \cdots s_i^{Z_i(m)} \exp\left\{ -\sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} \Big| T_N = n \right]$$
$$= \frac{D_i}{2^i} \frac{g_{i+1,N}}{g_{1,N}} \left(c_{i+1,N} + \sum_{l=i+1}^{N} \lambda_l a_{i+1,l} \right)^{-1+1/2^i}.$$

We see that, under the conditions of Theorem 4 there are no particles of types $1,2,...,i$ in the limit.

Theorem 5 *If $m \sim xn, x \in (0,1)$, then*

$$\lim_{m\to\infty} \mathbf{E}\left[s_1^{Z_1(m)} \cdots s_{N-1}^{Z_{N-1}(m)} \exp\left\{ -\lambda_N \frac{Z_N(m)}{b_N n} \right\} \Big| T_N = n \right]$$
$$= \frac{1}{(1+(1-x)\lambda_N)^{1-\gamma_1}} \frac{1}{(1+\lambda_N x(1-x))^{1+\gamma_1}}.$$

It follows from Theorem 5 that at the final stage of the development the population contains particles of type N only.

We note that Theorems 2-5 may be considered as the statements describing the distribution of the number of vertices in the layers of certain classes simply generated random trees having a fixed hight (see [7]). The vertices of such trees are colored by one of N colors labelled by numbers 1 through N, and the numbers of the colors are monotone decreasing from the leaves to the root. The reader may find a more detailed information about the properties of simply generated trees and their connection with branching processes in a recent survey [3].

2 Preliminary arguments

In the sequel we denote by $\varepsilon_i(n), \varepsilon_i(n;m), i = 1,2,...$ some functions vanishing as $n \to \infty$. These function may be not necessary the same in different formulas.

Lemma 6 *Let A, B, α and β be positive numbers, $\alpha > \beta, \beta \in (0,1)$, and let $\Delta_n, n = 0,1,2,...$ be a sequence nonnegative numbers meeting the recurrent relationships*

$$\Delta_0 = 0, \Delta_n = \frac{A}{n^\alpha}(1+\varepsilon_1(n)) + \Delta_{n-1}\left(1 - \frac{B}{n^\beta}(1+\varepsilon_2(n))\right), \quad n = 1,2,.... \tag{12}$$

Then

$$\lim_{n\to\infty} n^{\alpha-\beta}\Delta_n = \frac{A}{B}.$$

Proof. Set $\gamma = \alpha - \beta$ and write $\Delta_n, n \geq 2$ in the form

$$\Delta_n = \frac{A}{B}\frac{1}{n^\gamma}\left(1 + \frac{\psi(n)}{\log n}\right). \tag{13}$$

Since

$$\frac{1}{(n-1)^\gamma} = \frac{1}{n^\gamma}\left(1 + \frac{\gamma}{n}(1 + \varepsilon_3(n))\right),$$

and

$$\frac{1}{\log(n-1)} = \frac{1}{\log n}\left(1 + \frac{1}{n\log n}(1 + \varepsilon_4(n))\right), \tag{14}$$

(12) takes the from

$$\frac{A}{B}\frac{1}{n^\gamma}\left(1 + \frac{\psi(n)}{\log n}\right) = \frac{A}{n^\alpha}(1 + \varepsilon_1(n))$$
$$+\frac{A}{B}\frac{1}{n^\gamma}\left(1 + \frac{\psi(n-1)}{\log(n-1)}\right)\left(1 + \frac{\gamma}{n}(1 + \varepsilon_3(n))\right)\left(1 - \frac{B}{n^\beta}(1 + \varepsilon_2(n))\right),$$

which, after evident transformations based on the condition $\beta < 1$ and the equalities

$$\frac{A}{B}\frac{1}{n^\gamma}\left(1 + \frac{\gamma}{n}(1 + \varepsilon_3(n))\right)\left(1 - \frac{B}{n^\beta}(1 + \varepsilon_2(n))\right)$$
$$= \frac{A}{B}\frac{1}{n^\gamma}\left(1 - \frac{B}{n^\beta}(1 + \varepsilon_4(n))\right) = \frac{A}{B}\frac{1}{n^\gamma} - \frac{A}{n^\alpha} + \frac{\varepsilon_5(n)}{n^\alpha},$$

leads to

$$\frac{\psi(n)}{\log n} = \frac{\varepsilon_5(n)}{n^\beta} + \frac{\psi(n-1)}{\log(n-1)}\left(1 - \frac{B}{n^\beta}(1 + \varepsilon_6(n))\right)$$

or, on account of (14),

$$\psi(n) = \frac{\varepsilon_5(n)\log n}{n^\beta} + \psi(n-1)\left(1 - \frac{B}{n^\beta}(1 + \varepsilon_6(n))\right).$$

We show that

$$\limsup_{n\to\infty}\frac{|\psi(n)|}{\log n} = 0. \tag{15}$$

If this is not the case, then there exists such a subsequence $n_k \to \infty$ as $k \to \infty$, that

$$\limsup_{k\to\infty}\frac{|\psi(n_k)|}{\log n_k} = c > 0. \tag{16}$$

Assume that $\psi(n_k) \to \infty$. Let k be such that

$$\psi(n_k) = \max_{1\leq n\leq n_k}\psi(n).$$

Then (to simplify notation we agree to write $n_k = n$)

$$\psi(n) \leq \frac{\varepsilon_5(n)\log n}{n^\beta} + \psi(n)\left(1 - \frac{B}{n^\beta}(1+\varepsilon_6(n))\right)$$

or

$$B(1+\varepsilon_6(n))\,\psi(n) \leq \varepsilon_5(n)\log n. \tag{17}$$

Assume now that $\psi(n_k) \to -\infty$. Let k be such that

$$\psi(n_k) = \min_{1\leq n\leq n_k} \psi(n).$$

Then (to simplify notation we agree to write $n_k = n$)

$$\psi(n) \geq \frac{\varepsilon_5(n)\log n}{n^\beta} + \psi(n)\left(1 - \frac{B}{n^\beta}(1+\varepsilon_6(n))\right)$$

or

$$B(1+\varepsilon_6(n))\,\psi(n) \geq \varepsilon_5(n)\log n. \tag{18}$$

Clearly, the combination of (17) and (18) contradicts (16). This proves (15).

It follows from the obtained estimate and (13) that

$$\Delta_n \sim \frac{A}{B}\frac{1}{n^{\alpha-\beta}}, \quad n \to \infty.$$

Lemma 6 is proved.

We use the symbols $\mathbf{P}_i$ and $\mathbf{E}_i$ to denote the probability and expectation calculated under the condition that a branching process is initiated at moment $n = 0$ by a single particle of type i. Sometimes we write $\mathbf{P}$ and $\mathbf{E}$ for $\mathbf{P}_1$ and $\mathbf{E}_1$, respectively.

Denote by

$$b_{ikl}(n) = \mathbf{E}_i\left[Z_k(n)Z_l(n) - \delta_{kl}Z_l(n)\right] \tag{19}$$

the second moments of the components of the process $\mathbf{Z}(n)$. Let $b_{ikl} = b_{ikl}(1)$.

For any vector $\mathbf{s} = (s_1, ..., s_p)$ (the dimension will usually be clear from the context), and any vector $\mathbf{k} = (k_1.....k_p)$ with integer valued components define

$$\mathbf{s}^{\mathbf{k}} = s_1^{k_1} ... s_p^{k_p}.$$

Further, let $\mathbf{1} = (1, ..., 1)$ be a vector of units. Sometimes it will be convenient to write $\mathbf{1}^{(i)}$ for the $i-$dimensional vector with all components equal to one.

Let

$$H_n^{(i,N)}(\mathbf{s}) = \mathbf{E}_i\left[\mathbf{s}^{\mathbf{Z}(n)}\right] = \mathbf{E}_i\left[s_i^{Z_i(n)} ... s_N^{Z_N(n)}\right]$$

be the probability generating functions for the process $\mathbf{Z}(n)$ given the process is initiated by a single particle of type $i \in \{1, 2, ..., N\}$ at moment 0. Clearly (see (1)), $H_1^{(i,N)}(\mathbf{s}) = h^{(i,N)}(\mathbf{s})$ for any $i \in \{1, ..., N\}$. Denote

$$Q_n^{(i,N)}(\mathbf{s}) = 1 - H_n^{(i,N)}(\mathbf{s}), \quad Q_n^{(i,N)} = 1 - H_n^{(i,N)}(\mathbf{0})$$

and let

$$\mathbf{H}_n(\mathbf{s}) = (H_n^{(1,N)}(\mathbf{s}), ..., H_n^{(N,N)}(\mathbf{s})), \ \mathbf{Q}_n(\mathbf{s}) = (Q_n^{(1,N)}(\mathbf{s}), ..., Q_n^{(N,N)}(\mathbf{s})).$$

The starting point of our arguments is the following theorem being a simplified combination of the respective results from [1] and [2]:

Theorem 7 *Let $\mathbf{Z}(n), n = 0, 1, ...$ be a decomposable critical branching process meeting the conditions (2), (3), (4) and (5). Then $m_{j,j}(n) = 1$ and, as $n \to \infty$*

$$\begin{aligned} m_{i,j}(n) &\sim a_{i,j} n^{j-i}, \ i < j, &(20)\\ b_{jpq}(n) &= a_{jpq} n^{p+q-2j+1} + o\left(n^{p+q-2j+1}\right), \ j \le \min(p,q), &(21)\end{aligned}$$

where $a_{i,j}$ are the same as in (11) and a_{jpq} are nonnegative constants known explicitly (see [2], Theorem 1).

In addition (see [1], Theorem 1), as $n \to \infty$

$$Q_n^{(i,N)} = 1 - H_n^{(i,N)}(\mathbf{0}) = \mathbf{P}_i(\mathbf{Z}(n) \ne \mathbf{0}) \sim c_{i,N} n^{-1/2^{N-i}}, \tag{22}$$

where $c_{i,N}$ are the same as in (9), and for $\lambda_l \ge 0$, $l = 1, 2, ..., N$,

$$\begin{aligned} &\lim_{m\to\infty} m\left(1 - \mathbf{E}\left[\exp\left\{-\sum_{l=1}^{N} \lambda_l \frac{Z_l(m)}{m^l}\right\}\right]\right) \\ &\qquad = \lim_{m\to\infty} m Q_m^{(1,N)}\left(e^{-\lambda_1/m}, e^{-\lambda_2/m^2}, ..., e^{-\lambda_N/m^N}\right) \\ &\qquad = \Phi_1(\lambda_1, \lambda_2, ..., \lambda_N). \end{aligned} \tag{23}$$

We need also the following Yaglom-type limit theorem proved in [16] and complementing Theorem 7.

Theorem 8 *If the conditions of Theorem 7 are valid, then for any $\lambda > 0$*

$$\lim_{n\to\infty} \mathbf{E}_1\left[\exp\left\{-\lambda \frac{Z_N(n)}{b_N n}\right\} \Big| \mathbf{Z}(n) \ne \mathbf{0}\right] = 1 - \left(\frac{\lambda}{1+\lambda}\right)^{1/2^{N-1}}. \tag{24}$$

3 Proof of Theorem 1

The results of the previous section allow us to prove Theorem 1. For $N = i$ we have

$$\begin{aligned} \mathbf{P}\left(T_{NN} = n\right) &= H_n^{(N,N)}(0) - H_{n-1}^{(N,N)}(0) \\ &= h^{(N,N)}(H_{n-1}^{(N,N)}(0)) - h^{(N,N)}(H_{n-2}^{(N,N)}(0)) \\ &\le H_{n-1}^{(N,N)}(0) - H_{n-2}^{(N,N)}(0) = \mathbf{P}\left(T_{NN} = n-1\right). \end{aligned}$$

Since

$$\mathbf{P}\left(T_{NN} > n\right) = \sum_{k=n+1}^{\infty} \mathbf{P}\left(T_{NN} = k\right) \sim \frac{1}{b_N n} \tag{25}$$

as $n \to \infty$ and the sequence $\mathbf{P}(T_{NN} = k)$ is monotone, (25) and Corollary 2 in [15] imply as $n \to \infty$

$$\mathbf{P}(T_{NN} = n) \sim \frac{1}{b_N n^2},$$

proving Theorem 1 for $i = N$. Assume that the theorem is proved for all $i \in \{j+1, N\}$, where $1 < j+1 \le N$. Let us demonstrate that it is true for $i = j$.

To this aim we put

$$\varphi(t) = h^{(j,N)}(\mathbf{H}_{n-2}(\mathbf{0}) + t(\mathbf{H}_{n-1}(\mathbf{0}) - \mathbf{H}_{n-2}(\mathbf{0}))),\ 0 \le t \le 1.$$

Clearly that

$$\mathbf{P}(T_{jN} = n) = \varphi(1) - \varphi(0) = \varphi'(0) + \varphi''(\theta_n)/2, \tag{26}$$

where $\theta_n \in [0,1]$.

It is easy to check that

$$\begin{aligned}
\varphi'(0) &= \sum_{i=j}^{N} \frac{\partial h^{(j,N)}(\mathbf{H}_{n-2}(\mathbf{0}))}{\partial s_i}(H_{n-1}^{(i,N)}(\mathbf{0}) - H_{n-2}^{(i,N)}(\mathbf{0})) \\
&= (1+\varepsilon(n)) \sum_{i=j+1}^{N} m_{j,i}\mathbf{P}(T_{iN} = n-1) + \frac{\partial h^{(j,N)}(\mathbf{H}_{n-2}(\mathbf{0}))}{\partial s_j}\mathbf{P}(T_{jN} = n-1),
\end{aligned}$$

where by the induction assumption

$$\sum_{i=j+1}^{N} m_{j,i}\mathbf{P}(T_{iN} = n-1) = (1+\varepsilon_1(n))\frac{m_{j,j+1}g_{j+1,N}}{n^{1+\gamma_{j+1}}},$$

and, in view of (22)

$$\begin{aligned}
\frac{\partial h^{(j,N)}(\mathbf{H}_{n-2}(\mathbf{0}))}{\partial s_j} &= 1 - (1+\varepsilon_2(n))\sum_{k=j}^{N}\frac{\partial^2 h^{(j,N)}(\mathbf{1})}{\partial s_k \partial s_j}(1 - H_{n-2}^{(k,N)}(\mathbf{0})) \\
&= 1 - (1+\varepsilon_3(n))b_{jjj}(1 - H_{n-2}^{(j,N)}(\mathbf{0})) = 1 - (1+\varepsilon_4(n))\frac{c_{j,N}b_{jjj}}{n^{\gamma_j}}.
\end{aligned}$$

Further, for $\Theta_n = \mathbf{H}_{n-2}(\mathbf{0}) + \theta_n(\mathbf{H}_{n-1}(\mathbf{0}) - \mathbf{H}_{n-2}(\mathbf{0}))$ we have

$$\begin{aligned}
\varphi''(\theta_n) &= \sum_{k=j}^{N}\sum_{i=j}^{N}\frac{\partial h^{(j,N)}(\Theta_n)}{\partial s_k \partial s_i}(H_{n-1}^{(k,N)}(\mathbf{0}) - H_{n-2}^{(k,N)}(\mathbf{0}))(H_{n-1}^{(i,N)}(\mathbf{0}) - H_{n-2}^{(i,N)}(\mathbf{0})) \\
&= (1+\varepsilon_5(n))\sum_{k=j}^{N}\sum_{i=j}^{N} b_{jik}\mathbf{P}(T_{kN} = n-1)\,\mathbf{P}(T_{iN} = n-1) \\
&= (1+\varepsilon_5(n))b_{jjj}\mathbf{P}^2(T_{j,N} = n-1) + o\Big(\frac{1}{n^{1+\gamma_{j+1}}}\Big).
\end{aligned}$$

Substituting the obtained estimates in (26) and recalling that $b_j = b_{jjj}/2$, we get

$$\begin{aligned}\mathbf{P}\left(T_{jN}=n\right) &= \frac{m_{j,j+1}g_{j+1,N}}{n^{1+\gamma_{j+1}}}\left(1+\varepsilon_1(n)\right) \\ &\quad +\mathbf{P}\left(T_{jN}=n-1\right)\left(1-\frac{2b_j c_{j,N}}{n^{\gamma_j}}\left(1+\varepsilon_2(n)\right)\right).\end{aligned}$$

This representation, Lemma 6 and the equalities

$$\frac{m_{j,j+1}g_{j+1,N}}{2b_j} = \gamma_{j+1}\frac{m_{j,j+1}c_{j+1,N}}{2b_j} = \gamma_j c_{j,N}^2 = c_{j,N}g_{jN}$$

yield, as $n\to\infty$

$$\mathbf{P}\left(T_{jN}=n\right) \sim \frac{m_{j,j+1}g_{j+1,N}}{2b_j c_{j,N}}\frac{1}{n^{1+\gamma_{j+1}-\gamma_j}} = \frac{g_{j,N}}{n^{1+\gamma_j}}.$$

This proves Theorem 1 by induction.

4 Auxiliary lemmas

We prove in this section a number of statements about the asymptotic behavior, as $n\to\infty$ expectations of the form

$$\mathbf{E}\left[\exp\left\{-\sum_{l=1}^{N}\lambda_l\frac{Z_l(m)}{r(n,m)}\right\}\right]$$

and their derivatives with respect to the parameters $\lambda_l, l = 1, 2, ..., N$ depending on the rate of growth the parameter $m = m(n)$ to infinity and the form of scaling $r(n,m)$. We will show that the asymptotic behavior of the mentioned quantities is essentially different for the cases $m \ll n^{\gamma_1}, m \sim yn^{\gamma_i}, y > 0$, $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}, i = 1, 2, ..., N-1$ and $m \sim xn, x \in (0,1)$.

4.1 The case $m \sim yn^{\gamma_i}, y > 0, 1 \le i \le N-1$

Let

$$I_k(m) = I\{Z_1(m) + \cdots + Z_k(m) = 0\}$$

be the indicator of the event that particles of types $1, 2, ..., k$ are absent in the population at time m. Suppose that $I_0(m) = 1$.

The aim of the present subsection is to prove the following lemma.

Lemma 9 *If the asymptotic relation $m \sim yn^{\gamma_i}, y > 0$, is true for some $i \in \{1, ..., N-1\}$, then for any $j \in \{i, ..., N-1\}$ and any tuple $\lambda_l \geq 0, l = i, ..., N$*

$$\lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{(j-i+1)\gamma_i}} \mathbf{E}\left[Z_j(m) \exp\left\{ -\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}} \right\} I_{i-1}(m) \right]$$
$$= \lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{(j-i+1)\gamma_i}} \mathbf{E}\left[Z_j(m) \exp\left\{ -\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}} \right\} \right]$$
$$= D_{i-1} \frac{\partial}{\partial \lambda_j} \left(\frac{\Phi_i(\lambda_i y, \lambda_{i+1} y^2, ..., \lambda_N y^{N-i+1})}{y} \right)^{1/2^{i-1}}.$$

The desired statement will be a corollary of a number of lemmas the first of them looks as follows.

Lemma 10 *If the asymptotic relation $m \sim yn^{\gamma_i}, y > 0$, is true for some $i \in \{1, ..., N-1\}$, then for $\lambda_l \geq 0$, $l = i, ..., N$ and*

$$\mathbf{s}(i) = \left(\exp\left\{ -\frac{\lambda_i}{n^{\gamma_i}} \right\}, \exp\left\{ -\frac{\lambda_{i+1}}{n^{2\gamma_i}} \right\}, ..., \exp\left\{ -\frac{\lambda_N}{n^{(N-i+1)\gamma_i}} \right\} \right)$$

we have

$$\lim_{n\to\infty} n^{\gamma_i} Q_m^{(i,N)}(\mathbf{s}(i)) = y^{-1} \Phi_i(\lambda_i y, \lambda_{i+1} y^2, ..., \lambda_N y^{N-i+1}).$$

Proof. Using (23) it is easy to check

$$\lim_{n\to\infty} n^{\gamma_1} Q_m^{(1,N)}(\mathbf{s}(i)) = y^{-1} \lim_{m\to\infty} m Q_m^{(1,N)}(\mathbf{s}(i))$$
$$= y^{-1} \lim_{m\to\infty} m \left(1 - \mathbf{E}\left[\exp\left\{ -\sum_{l=1}^{N} \lambda_l \frac{Z_l(m)}{n^{l\gamma_1}} \right\} \right] \right)$$
$$= y^{-1} \lim_{m\to\infty} m \left(1 - \mathbf{E}\left[\exp\left\{ -\sum_{l=1}^{N} \lambda_l y^l \frac{Z_l(m)}{m^l} \right\} \right] \right)$$
$$= y^{-1} \Phi_1(\lambda_1 y, \lambda_2 y^2, ..., \lambda_N y^N),$$

which proves the lemma for $i = 1$. The cases when $i \in \{2, ..., N-1\}$ may be considered in a similar way.

Lemma 10 is proved.

Lemma 11 *If $m \sim yn^{\gamma_j}$, $y > 0$, and $1 \leq i < j \leq N$, then for $\lambda_l \geq 0$, $l = j, ..., N$*

$$\lim_{n\to\infty} n^{(j-i+1)\gamma_i} \left(1 - \mathbf{E}_j\left[\exp\left\{ -\sum_{l=j}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}} \right\} \right] \right) = \sum_{l=j}^{N} \lambda_l y^{l-j} a_{j,l}.$$

Proof. Set

$$s_l = \exp\{-\frac{\lambda_l}{n^{(l-i+1)\gamma_i}}\},\ l = i, ..., N,$$

and consider the case $i = 1$ only, since the proof for $i \in \{2, ..., N-1\}$ requires only minor changes.

Clearly,

$$0 \le \sum_{l=j}^{N} (1 - s_l)\, \mathbf{E}_j\, [Z_l(m)] - Q_m^{(j,N)}(\mathbf{s}) \le \sum_{p,q=j}^{N} (1 - s_p)\,(1 - s_q)\, \mathbf{E}_j\, [Z_p(m) Z_q(m)]\,.$$

By (20) we have as $n \to \infty$

$$\sum_{l=j}^{N} (1 - s_l) \mathbf{E}_j Z_l(m) \sim \sum_{l=j}^{N} \lambda_l \frac{\mathbf{E}_j Z_l(m)}{n^{l\gamma_1}}$$
$$= \frac{1}{n^{j\gamma_1}} \sum_{l=j}^{N} \lambda_l y^{l-j} \frac{\mathbf{E}_j Z_l(m)}{m^{l-j}} \sim \frac{1}{n^{j\gamma_1}} \sum_{l=j}^{N} \lambda_l y^{l-j} a_{j,l}. \quad (27)$$

Further, in view of (21)

$$\sum_{p,q=j}^{N} \lambda_p \lambda_q \frac{\mathbf{E}_j\, [Z_p(m) Z_q(m)]}{n^{p\gamma_1} n^{q\gamma_1}} = \frac{1}{n^{j\gamma_1}} \sum_{p,q=j}^{N} \lambda_p \lambda_q \frac{b_{jpq}\,(m)}{n^{(p-j)\gamma_1} n^{q\gamma_1}}$$
$$= \frac{1}{n^{j\gamma_1}} \sum_{p,q=j}^{N} \lambda_p \lambda_q \frac{a_{jpq} m^{p+q-2j+1}}{n^{(p-j)\gamma_1} n^{q\gamma_1}} + \varepsilon_1(n) \frac{1}{n^{j\gamma_1}} \sum_{p,q=j}^{N} \frac{m^{p+q-2j+1}}{n^{(p-j)\gamma_1} n^{q\gamma_1}}$$
$$\le C \frac{1}{n^{j\gamma_1}} \sum_{p,q=j}^{N} \frac{n^{\gamma_1(p+q-2j+1)}}{n^{(p-j)\gamma_1} n^{q\gamma_1}} = o\left(\frac{1}{n^{j\gamma_1}}\right).$$

The obtained estimates prove the lemma.

Let $\eta_{r,j}\,(k,l)$ be the number of type j daughter particles of the $l-$th particle of type r, belonging to the $k-$th generation and let

$$W_{p,i,j} = \sum_{r=p}^{i} \sum_{k=0}^{T_i} \sum_{q=1}^{Z_r(k)} \eta_{r,j}\,(k,q)$$

be the total number of type $j \ge i+1$ daughter particles generated by all the particles of types $p, p+1, ..., i$ ever born in the process given that the process is initiated at time $n = 0$ by a single particle of type $p \le i$. Finally, put

$$W_{p,i} = \sum_{j=i+1}^{N} W_{p,i,j} = \sum_{j=i+1}^{N} \sum_{r=p}^{i} \sum_{k=0}^{T_i} \sum_{q=1}^{Z_r(k)} \eta_{r,j}\,(k,q)\,.$$

Lemma 12 *(see [16], Lemma 1). Let Hypothesis A be valid. Then, as $\lambda \downarrow 0$*

$$1-\mathbf{E}\left[e^{-\lambda W_{1,i,i+1}}|\mathbf{Z}(0)=\mathbf{e}_1\right]\sim D_i\lambda^{1/2^i} \tag{28}$$

and there exists a constant $F_i>0$ such that

$$1-\mathbf{E}\left[e^{-\lambda W_{1,i}}|\mathbf{Z}(0)=\mathbf{e}_1\right]\sim F_i\lambda^{1/2^i}. \tag{29}$$

Basing on Lemmas 11 and 12 we prove the following statement.

Lemma 13 *If $m\sim yn^{\gamma_j}$, $y>0$, for some $i\in\{1,2,...,N-1\}$, then for $\lambda_l\geq 0$, $l=i,...,N$*

$$\begin{aligned}\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}&\left[\left(1-\exp\left\{-\sum_{l=i}^{N}\lambda_l\frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right)I_{i-1}(m)\right]\\ &=\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[1-\exp\left\{-\sum_{l=i}^{N}\lambda_l\frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right]\\ &=D_{i-1}\left(\frac{\Phi_i(\lambda_i y,\lambda_{i+1}y^2,...,\lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}}.\end{aligned} \tag{30}$$

Proof. For $i=1$ the statement of the lemma is a particular case of Lemma 10. Thus, we assume now that $i\in\{2,3,...,N-1\}$. According to (22) for $m\sim yn^{\gamma_i}$ the following relations are valid:

$$\begin{aligned}\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}&\left[\left(1-\exp\left\{-\sum_{l=i}^{N}\lambda_l\frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right)(1-I_{i-1}(m))\right]\\ &\leq\lim_{n\to\infty} n^{\gamma_1}\mathbf{P}(T_{i-1}>m)=\lim_{n\to\infty}\frac{c_{1,i-1}n^{1/2^{N-1}}}{(yn^{1/2^{N-i}})^{1/2^{i-2}}}=0.\end{aligned}$$

Therefore, to prove the lemma it is sufficient to show the validity of the second equality in (30) only. Recalling (22) once more, we have, as $n\to\infty$,

$$\mathbf{P}(T_{1,i-1}>n^{3\gamma_{i-2}})\sim c_{1,i-1}n^{-3\gamma_{i-2}/2^{i-2}}=c_{1,i-1}n^{-3\gamma_1/2}=o(n^{-\gamma_1}).$$

Thus,

$$\begin{aligned}Q_m^{(1,N)}(\mathbf{s}) &= \mathbf{E}\left[1-s_1^{Z_1(m)}s_2^{Z_2(m)}...s_N^{Z_N(m)}\right]\\ &= \mathbf{E}\left[\left(1-s_i^{Z_i(m)}...s_N^{Z_N(m)}\right);T_{i-1}\leq n^{3\gamma_{i-2}}\right]+o(n^{-\gamma_1})\\ &= 1-H_m^{(1,N)}\left(\mathbf{1}^{(i-1)},s_i,...,s_N\right)+o(n^{-\gamma_1}).\end{aligned}$$

It is not difficult to check that for our decomposable branching process

$$
\begin{aligned}
&H_m^{(1,N)}\left(\mathbf{1}^{(i-1)}, s_i, ..., s_N\right) \\
&= \mathbf{E}\left[\prod_{k=0}^{m-1}\prod_{r=1}^{i-1}\prod_{l=1}^{Z_r(k)}\prod_{j=i}^{N}\left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)}\right] \\
&= \mathbf{E}\left[\prod_{k=0}^{m-1}\prod_{r=1}^{i-1}\prod_{l=1}^{Z_r(k)}\prod_{j=i}^{N}\left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)}; T_{i-1}\le n^{3\gamma_{i-2}}\right] + o(n^{-\gamma_1}).
\end{aligned}
$$

Observing that $\lim_{m\to\infty} H_{m-k}^{(j,N)}(\mathbf{s}) = 1$ for $k \le T_{i-1} \le n^{3\gamma_{i-2}} = o(m)$ and $j \ge i+1$, we conclude that, on the set $T_{i-1} \le n^{3\gamma_{i-2}}$

$$
\begin{aligned}
&\prod_{k=0}^{m-1}\prod_{r=1}^{i-1}\prod_{l=1}^{Z_r(k)}\prod_{j=i}^{N}\left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)} \\
&\quad = \exp\left\{-\sum_{r=1}^{i-1}\sum_{k=0}^{T_{i-1}}\sum_{l=1}^{Z_r(k)}\sum_{j=i}^{N}\eta_{r,j}\left(k,l\right)Q_{m-k}^{(j,N)}(\mathbf{s})(1+o(1))\right\}.
\end{aligned}
$$

Note now that according to (23) for $m \sim yn^{\gamma_i}$,

$$
s_l = \exp\left\{-\frac{\lambda_l}{n^{(l-i+1)\gamma_i}}\right\}, l = i, ..., N
$$

and $k = o(m)$ the following relations are valid:

$$
\begin{aligned}
\lim_{n\to\infty} n^{\gamma_i} Q_{m-k}^{(i,N)}(\mathbf{s}) &= y^{-1}\lim_{m\to\infty} m\left(1 - \mathbf{E}_i\left[\exp\left\{-\sum_{l=i}^{N}\lambda_l y^{l-i+1}\frac{Z_l(m)}{m^{l-i+1}}\right\}\right]\right) \\
&= y^{-1}\Phi_i(\lambda_i y, \lambda_{i+1}y^2, ..., \lambda_N y^{N-i+1}),
\end{aligned}
$$

while by Lemma 11 we have for $j > i$

$$
\lim_{n\to\infty} n^{(j-i+1)\gamma_i} Q_{m-k}^{(j,N)}(\mathbf{s}) = \sum_{l=j}^{N}\lambda_l y^{l-j}a_{j,l}. \tag{31}
$$

Hence it follows that if the condition $T_{i-1} \le \sqrt{mn^{\gamma_{i-1}}} = o(n^{\gamma_i})$ is valid, then

$$
\begin{aligned}
&\sum_{r=1}^{i-1}\sum_{k=0}^{T_{i-1}}\sum_{l=1}^{Z_r(k)}\sum_{j=i}^{N}\eta_{r,j}\left(k,l\right)Q_{m-k}^{(j,N)}(\mathbf{s}) \\
&\quad = (1+o(1))\sum_{j=i}^{N}Q_m^{(j,N)}(\mathbf{s})\sum_{r=1}^{i-1}\sum_{k=0}^{T_{i-1}}\sum_{l=1}^{Z_r(k)}\eta_{r,j}\left(k,l\right) \\
&\quad = (1+o(1))\sum_{j=i}^{N}W_{1,i-1,j}Q_m^{(j,N)}(\mathbf{s}).
\end{aligned}
$$

Further, for $m \sim yn^{\gamma_i}$

$$W_{1,i-1,i}Q_m^{(i,N)}(\mathbf{s}) = (1+o(1))\, W_{1,i-1,i}y^{-1}\Phi_i(\lambda_i y, \lambda_{i+1}y^2, ..., \lambda_N y^{N-i+1})n^{-\gamma_i}, \tag{32}$$

while by (31)

$$\sum_{j=i+1}^{N} W_{1,i-1,j}Q_m^{(j,N)}(\mathbf{s}) = O\left(\sum_{j=i+1}^{N} W_{1,i-1,j}n^{-(j-i+1)\gamma_i}\right) = O(n^{-2\gamma_i}W_{1,i-1}).$$

Using Lemma 12 we conclude that

$$\begin{aligned} 0 \quad \leq \quad & \mathbf{E}\left[\exp\left\{-(1+o(1))W_{1,i-1,i}Q_m^{(i,N)}(\mathbf{s})\right\}\right] \\ & -\mathbf{E}\left[\exp\left\{-(1+o(1))W_{1,i-1,i}Q_m^{(i,N)}(\mathbf{s}) - O(n^{-2\gamma_i}W_{1,i-1})\right\}\right] \\ & \leq 1 - \mathbf{E}\left[\exp\left\{-O(n^{-\gamma_{i+1}}W_{1,i-1})\right\}\right] = O\left(\left(n^{-\gamma_{i+1}}\right)^{1/2^{i-1}}\right) = O\left(n^{-\gamma_2}\right). \end{aligned}$$

As a result on account of (32) we have for $m \sim yn^{\gamma_i}$

$$\begin{aligned} Q_m^{(1,N)}(\mathbf{s}) &= 1 - H_m^{(1,N)}\left(\mathbf{1}^{(i-1)}, s_i, ..., s_N\right) + o(n^{-\gamma_1}) \\ &= 1 - \mathbf{E}\left[\exp\left\{-(1+o(1))W_{1,i-1,i}Q_m^{(i,N)}(\mathbf{s})\right\}\right] + o(n^{-\gamma_1}) \\ &= D_{i-1}\left(Q_m^{(i,N)}(\mathbf{s})\right)^{1/2^{i-1}} + o(n^{-\gamma_1}), \end{aligned}$$

as required.

The lemma is proved.

Proof of Lemma 9. Recalling Lemma 13 and using the relation $\log(1-x) = -x + o(x), x \downarrow 0$, we have

$$\begin{aligned} & \lim_{n\to\infty} \mathbf{E}^{n^{\gamma_1}}\left[\exp\left\{-\sum_{l=i}^{N}\lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right] \\ &= \lim_{n\to\infty} \exp\left\{n^{\gamma_1}\log \mathbf{E}\left[\exp\left\{-\sum_{l=i}^{N}\lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right]\right\} \\ &= \exp\left\{-\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[1-\exp\left\{-\sum_{l=i}^{N}\lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right]\right\} \\ &= \exp\left\{-D_{i-1}\left(\frac{\Phi_i(\lambda_i y, \lambda_{i+1}y^2, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}}\right\}. \end{aligned} \tag{33}$$

Since the prelimiting function in $N-i+1$ complex variables $\lambda_i, ..., \lambda_N$ is analytical and bounded in the domain $\{Re\lambda_l > 0, l = i, i+1, ..., N\}$:

$$\left|\mathbf{E}^{n^{\gamma_1}}\left[\exp\left\{-\sum_{l=i}^{N}\lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right]\right| \leq 1$$

and converges for the real-valued $\lambda_l > 0, l = i, i+1, ..., N$, it follows by the Vitali and Weierstrass theorems that the limiting function is analytical in the domain $\{Re\lambda_l > 0, l = i, i+1, ..., N\}$ and, in addition, the derivative of the prelimiting function with resect to any variable converges to the respective derivative of the limiting function. Hence, on account of the equality

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right] = 1$$

it is not difficult to deduce that

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{(j-i+1)\gamma_i}} \mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right] \mathbf{E}^{n^{\gamma_1}}\left[\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right] \\ = -\frac{\partial}{\partial\lambda_j} \exp\left\{-D_{i-1}\left(\frac{\Phi_i(\lambda_i y, \lambda_{i+1}y^2, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}}\right\},\end{aligned}$$

or, in view of (33)

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{(j-i+1)\gamma_i}} \mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right] \\ = D_{i-1}\frac{\partial}{\partial\lambda_j}\left(\frac{\Phi_i(\lambda_i y, \lambda_{i+1}y^2, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}}.\end{aligned}$$

The first part of Lemma 9 is proved.

To prove the second part it is necessary, basing on the representation

$$\begin{aligned}&\mathbf{E}^{n^{\gamma_1}}\left[1-\left(1-\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right) I_{i-1}(m)\right] \\ = \quad &(1+o(1))\exp\left\{n^{\gamma_1}\mathbf{E}\left[\left(1-\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\}\right) I_{i-1}(m)\right]\right\},\end{aligned}$$

to repeat almost literally the arguments used earlier.

The lemma is proved.

4.2 The case $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}, 1 \le i \le N-1$

The aim of the present subsection is to check the validity of the following statement:

Lemma 14 *If $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ for some $i \in \{1, 2, ..., N-1\}$, then, for any*

$j \in \{i, ..., N-1\}$ *and* $\lambda_l \geq 0, l = i, ..., N$

$$\lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{\gamma_{i+1}} m^{j-i-1}} \mathbf{E}\left[Z_j(m) \exp\left\{ -\sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} I_i(m) \right]$$

$$= \lim_{n\to\infty} \frac{n^{\gamma_1}}{n^{\gamma_{i+1}} m^{j-i-1}} \mathbf{E}\left[Z_j(m) \exp\left\{ -\sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} \right]$$

$$= \frac{D_i a_{i+1,j}}{2^i} \left(\sum_{l=i+1}^{N} \lambda_l a_{i+1,l} \right)^{-1+1/2^i}.$$

The needed result will be a corollary of a number of auxiliary statements.

Lemma 15 *If* $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ *for some* $i \in \{1, 2, ..., N-1\}$*, then, for* $j \geq i+1$ *and* $\lambda_l \geq 0$, $l = j, ..., N$

$$\lim_{n\to\infty} n^{\gamma_{i+1}} m^{j-i-1} \left(1 - \mathbf{E}_j \left[\exp\left\{ -\sum_{l=j}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} \right] \right) = \sum_{l=j}^{N} \lambda_l a_{j,l}.$$

Proof. Put

$$s_l = \exp\left\{ -\frac{\lambda_l}{n^{\gamma_{i+1}} m^{l-i-1}} \right\},\ l = i+1, ..., N. \tag{34}$$

It is not difficult to check that, under the choice of variables

$$0 \leq \sum_{l=j}^{N} \lambda_l \frac{\mathbf{E}_j [Z_l(m)]}{m^{l-j}} - n^{\gamma_{i+1}} m^{j-i-1} \left(1 - \mathbf{E}_j \left[\prod_{l=j}^{N} s_l^{Z_l(m)} \right] \right)$$
$$\leq \frac{m^{j-i-1}}{n^{\gamma_{i+1}}} \sum_{p,q=j}^{N} \lambda_p \lambda_q \frac{\mathbf{E}_j [Z_p(m) Z_q(m)]}{m^{p-i-1} m^{q-i-1}}.$$

Using this inequality and the relations

$$\sum_{l=j}^{N} \lambda_l \frac{\mathbf{E}_j [Z_l(m)]}{m^{l-j}} \sim \sum_{l=j}^{N} \lambda_l a_{j,l}$$

and

$$\begin{aligned} \sum_{p,q=j}^{N} \lambda_p \lambda_q \frac{\mathbf{E}_j [Z_p(m) Z_q(m)]}{n^{2\gamma_{i+1}} m^{p-i-1} m^{q-i-1}} &\leq \frac{C}{n^{2\gamma_{i+1}}} \sum_{p,q=j+2}^{N} \frac{m^{p+q-2j+1}}{m^{p-i-1} m^{q-i-1}} \\ &\leq \frac{CN^2}{n^{2\gamma_{i+1}}} m^{2(i-j)+3} = \frac{CN^2}{n^{\gamma_{i+1}} m^{j-i-1}} \times \frac{m^{i-j+2}}{n^{\gamma_{i+1}}} \\ &\leq \frac{CN^2}{n^{\gamma_{i+1}} m^{j-i-1}} \times \frac{m}{n^{\gamma_{i+1}}} = o\left(\frac{1}{n^{\gamma_{i+1}} m^{j-i-1}} \right), \end{aligned}$$

following from Theorem 7, it is not difficult to demonstrate the validity of the lemma.

Lemma 16 *If $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ for some $i \in \{1, ..., N-1\}$, then for $\lambda_l \geq 0$, $l = i+1, ..., N$*

$$\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[\left(1-\exp\left\{-\sum_{l=i+1}^{N}\lambda_l\frac{Z_l(m)}{n^{\gamma_{i+1}}m^{l-i-1}}\right\}\right)I_i\left(m\right)\right]$$
$$=\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[1-\exp\left\{-\sum_{l=i+1}^{N}\lambda_l\frac{Z_l(m)}{n^{\gamma_{i+1}}m^{l-i-1}}\right\}\right]$$
$$=D_i\left(\sum_{l=i+1}^{N}\lambda_l a_{i+1,l}\right)^{1/2^i}.$$

Proof. As before, it is sufficient to show the validity of the second equality. Similarly to the arguments used earlier in the proof of Lemma 13, we have

$$\mathbf{P}(T_i > m) \sim c_{1,i}m^{-2^{-(i-1)}} = o(n^{-\gamma_1}).$$

Thus,

$$\begin{aligned}
Q_m^{(1,N)}(\mathbf{s}) &= \mathbf{E}\left[1-\prod_{j=1}^{N}s_j^{Z_j(m)}\right] \\
&= \mathbf{E}\left[\left(1-\prod_{j=i+1}^{N}s_j^{Z_j(m)}\right);T_i\leq m\right]+o(n^{-\gamma_1}) \\
&= 1-H_m^{(1,N)}\left(\mathbf{1}^{(i)},s_{i+1},...,s_N\right)+o(n^{-\gamma_1}).
\end{aligned}$$

Further,

$$\begin{aligned}
&H_m^{(1,N)}\left(\mathbf{1}^{(i)},s_{i+1},...,s_N\right) \\
&=\mathbf{E}\left[\prod_{k=0}^{m-1}\prod_{r=1}^{i}\prod_{l=1}^{Z_r(k)}\prod_{j=i+1}^{N}\left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)}\right] \\
&=\mathbf{E}\left[\prod_{k=0}^{m-1}\prod_{r=1}^{i}\prod_{l=1}^{Z_r(k)}\prod_{j=i+1}^{N}\left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)};T_i\leq\sqrt{mn^{\gamma_i}}\right] \\
&\quad+O\left(\mathbf{P}\left(T_i>\sqrt{mn^{\gamma_i}}\right)\right).
\end{aligned}$$

Observing that $\lim_{m\to\infty} H_{m-k}^{(j,N)}(\mathbf{s}) = 1$ for $k \leq T_i \leq \sqrt{mn^{\gamma_i}} = o(m)$ and $j \geq i+1$, we conclude that, on the set $T_i \leq \sqrt{mn^{\gamma_i}}$

$$\prod_{k=0}^{m-1} \prod_{r=1}^{i} \prod_{l=1}^{Z_r(k)} \prod_{j=i+1}^{N} \left(H_{m-k}^{(j,N)}(\mathbf{s})\right)^{\eta_{r,j}(k,l)}$$
$$= \exp\left\{-\sum_{r=1}^{i}\sum_{k=0}^{T_i}\sum_{l=1}^{Z_r(k)}\sum_{j=i+1}^{N} \eta_{r,j}(k,l)\, Q_{m-k}^{(j,N)}(\mathbf{s})(1+o(1))\right\}.$$

Lemma 15 and the estimate $m \ll n^{\gamma_{i+1}}$ give for $k = o(m)$ and $s_l, l = i+1, ..., N$, from (34) :

$$Q_{m-k}^{(j,N)}(\mathbf{s}) \sim Q_m^{(j,N)}(\mathbf{s}) \sim \frac{1}{n^{\gamma_{i+1}} m^{j-i-1}} \sum_{l=j}^{N} \lambda_l a_{j,l}. \tag{35}$$

Hence it follows that the relations

$$\sum_{r=1}^{i}\sum_{k=0}^{T_i}\sum_{l=1}^{Z_r(k)}\sum_{j=i+1}^{N} \eta_{r,j}(k,l)\, Q_{m-k}^{(j,N)}(\mathbf{s})$$
$$= (1+o(1)) \sum_{j=i+1}^{N} Q_m^{(j,N)}(\mathbf{s}) \sum_{r=1}^{i}\sum_{k=0}^{T_i}\sum_{l=1}^{Z_r(k)} \eta_{r,j}(k,l)$$
$$= (1+o(1)) \sum_{j=i+1}^{N} W_{1,i,j} Q_m^{(j,N)}(\mathbf{s})$$
$$= (1+o(1)) W_{1,i,i+1} Q_m^{(i+1,N)}(\mathbf{s}) + O\left(Q_m^{(i+2,N)}(\mathbf{s}) W_{1,i}\right)$$

are valid on the set $T_i \leq \sqrt{mn^{\gamma_i}} = o(m) = o(n^{\gamma_{i+1}})$. Using the estimates

$$\begin{aligned}
0 \quad \leq \quad & \mathbf{E}\left[\exp\left\{-(1+o(1))W_{1,i,i+1}Q_m^{(i+1,N)}(\mathbf{s})\right\}\right] \\
& -\mathbf{E}\left[\exp\left\{-(1+o(1))W_{1,i,i+1}Q_m^{(i+1,N)}(\mathbf{s}) - O\left(Q_m^{(i+2,N)}(\mathbf{s})W_{1,i}\right)\right\}\right] \\
& \quad \leq 1 - \mathbf{E}\left[\exp\left\{-O\left(Q_m^{(i+2,N)}(\mathbf{s})W_{1,i}\right)\right\}\right] = O\left(\left(\frac{1}{n^{\gamma_{i+1}} m}\right)^{1/2^i}\right) \\
& \quad = o\left(\left(\frac{1}{n^{\gamma_{i+1}} n^{\gamma_i}}\right)^{1/2^i}\right) = o\left(n^{-3\gamma_1/2}\right) = o\left(n^{-\gamma_1}\right),
\end{aligned}$$

following from (29) and (35), and recalling (28) we conclude that

$$\begin{aligned}
Q_m^{(1,N)}(\mathbf{s}) \quad &= \quad 1 - \exp\left\{-(1+o(1))W_{1,i,i+1} n^{-\gamma_{i+1}} \sum_{l=i+1}^{N} \lambda_l a_{i+1,l}\right\} + o\left(n^{-\gamma_1}\right) \\
&= \quad D_i \left(\sum_{l=i+1}^{N} \lambda_l a_{i+1,l}\right)^{1/2^i} n^{-\gamma_1} + o\left(n^{-\gamma_1}\right),
\end{aligned}$$

as required.

Proof of Lemma 14. To demonstrate the validity of Lemma 14 it is sufficient to recall Lemma 16 and to repeat (with evident changes) the arguments used to prove Lemma 9.

4.3 The case $m \ll n^{\gamma_1}$

Lemma 17 *If the parameters m and n tend to infinity in such a way that $m \ll n^{\gamma_1}$, then for any $j \in \{1, ..., N\}$ and $\lambda_l \geq 0$, $l = 1, ..., N$*

$$\lim_{m\to\infty} \frac{1}{m^{j-1}} \mathbf{E}\left[Z_j(m) \exp\left\{-\sum_{l=1}^{N} \lambda_l \frac{Z_l(m)}{m^l}\right\}\right] = \frac{\partial \Phi_1(\lambda_1, \lambda_2, ..., \lambda_N)}{\partial \lambda_j}.$$

Proof. Recalling (23) and repeating the arguments used to demonstrate Lemma 9, we see that

$$\begin{aligned} Q_m^{(1,N)}(\mathbf{s}) &= 1 - \exp\left\{-(1+o(1))W_{1,i,i+1}n^{-\gamma_{i+1}} \sum_{l=i+1}^{N} \lambda_l a_{i+1,l}\right\} + o\left(n^{-\gamma_1}\right) \\ &= D_i \left(\sum_{l=i+1}^{N} \lambda_l a_{i+1,l}\right)^{1/2^i} n^{-\gamma_1} + o\left(n^{-\gamma_1}\right), \end{aligned}$$

as required.

Note, that

$$\frac{\partial \Phi_1(\lambda_1, \lambda_2, ..., \lambda_N)}{\partial \lambda_1} |_{\lambda=\mathbf{0}} = 1. \tag{36}$$

5 Proof of the limit theorems

For $m < n$ introduce the functions

$$\Psi^{(i,N)}(m, n; \mathbf{s}) = \mathbf{E}_i \left[\mathbf{s}^{\mathbf{Z}(m)} I\left\{T_{iN} = n\right\}\right],$$

where $I\{A\}$ is the indicator of the event A. Our aim is to investigate the asymptotic behavior of the quantity

$$\mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m)} | T_{1N} = n\right] = \frac{\Psi^{(i,N)}(m, n; \mathbf{s})}{\mathbf{P}\left(T_{1N} = n\right)}$$

depending on the rate of growth of n and m to infinity. Clearly,

$$\begin{aligned} &\Psi^{(1,N)}(m, n; \mathbf{s}) \\ &= \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m)} \left(\prod_{l=1}^{N} \mathbf{P}_l^{Z_l(m)}\left(\mathbf{Z}(n-m) = \mathbf{0}\right) - \prod_{l=1}^{N} \mathbf{P}_l^{Z_l(m)}\left(\mathbf{Z}(n-m-1) = \mathbf{0}\right)\right)\right]. \end{aligned}$$

Using the formula

$$\prod_{l=1}^{N} X_l - \prod_{l=1}^{N} Y_l = \sum_{j=1}^{N} (X_j - Y_j) \prod_{l=1}^{j-1} Y_l \prod_{l=j+1}^{N} X_l,$$

where $\prod_{l=1}^{0} Y_l = \prod_{N+1}^{N} X_l = 1$, and setting

$$X_l = \mathbf{P}_l^{Z_l(m)} \left(\mathbf{Z}(n-m) = \mathbf{0}\right), \quad Y_l = \mathbf{P}_l^{Z_l(m)} \left(\mathbf{Z}(n-m-1) = \mathbf{0}\right),$$

we obtain

$$\Psi^{(1,N)}(m,n;\mathbf{s}) = \sum_{j=1}^{N} G_j\left(m,n;\mathbf{s}\right), \tag{37}$$

where

$$G_j\left(m,n;\mathbf{s}\right) = \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m)} \left(X_j - Y_j\right) \prod_{l=1}^{j-1} Y_l \prod_{l=j+1}^{N} X_l\right]. \tag{38}$$

We separately investigate the behavior of the functions $G_j\left(m,n;\mathbf{s}\right)$ under an appropriate choice of the relationship between m and n and an appropriate choice of the components of $\mathbf{s}$.

Let

$$x_l = \mathbf{P}_l\left(\mathbf{Z}(n-m) = \mathbf{0}\right), \quad y_l = \mathbf{P}_l\left(\mathbf{Z}(n-m-1) = \mathbf{0}\right). \tag{39}$$

Then

$$\begin{aligned} &\mathbf{E}\left[Z_j(m)\mathbf{s}^{\mathbf{Z}(m)} \left(x_j - y_j\right) y_j^{Z_j(m)} \prod_{l=1}^{j-1} Y_l \prod_{l=j+1}^{N} X_l\right] \\ &\quad \leq G_j\left(m,n;\mathbf{s}\right) \\ &\quad\quad \leq \mathbf{E}\left[Z_j(m)\mathbf{s}^{\mathbf{Z}(m)} \left(x_j - y_j\right) x_j^{Z_j(m)} \prod_{l=1}^{j-1} Y_l \prod_{l=j+1}^{N} X_l\right]. \end{aligned} \tag{40}$$

In view of the asymptotic relations (22) and Theorem 1, we have

$$\mathbf{P}_i\left(\mathbf{Z}(n) \neq \mathbf{0}\right) \sim \frac{c_{i,N}}{n^{\gamma_i}}, \quad \mathbf{P}\left(T_{iN} = n\right) \sim \frac{g_{i,N}}{n^{1+\gamma_i}}.$$

Since

$$\begin{aligned} X_l &= \exp\left\{-Z_l(m)\mathbf{P}_l\left(\mathbf{Z}\left(n-m\right) \neq \mathbf{0}\right)\left(1+\varepsilon_l\left(n,m\right)\right)\right\} \\ &= \left(1+\tilde{\varepsilon}_l\left(n,m\right)\right)\exp\left\{-c_{l,N}\frac{Z_l(m)}{(n-m)^{\gamma_l}}\right\}, \end{aligned}$$

$$\begin{aligned} Y_l &= \exp\left\{-Z_l(m)\mathbf{P}_l\left(\mathbf{Z}\left(n-m-1\right) \neq \mathbf{0}\right)\left(1+\varepsilon_l\left(n,m+1\right)\right)\right\} \\ &= \left(1+\tilde{\varepsilon}_l\left(n,m+1\right)\right)\exp\left\{-c_{l,N}\frac{Z_l(m)}{(n-m)^{\gamma_l}}\right\}, \end{aligned}$$

it follows that for $m \ll n$ and $s_l = \exp\{-\lambda_l/L_l(m)\}$, where the functions $L_l(m), l = 1, 2, ..., N$ will be selected later on depending on the range of m under consideration, it is necessary to investigate, for each $j = 1, 2, ..., N$ and up to negligible terms, the asymptotic behavior of the quantity

$$\begin{aligned} C_j(m,n) &= \mathbf{E}\left[Z_j(m)\mathbf{s}^{\mathbf{Z}(m)}\left(x_j - y_j\right) y_j^{Z_j(m)} \prod_{l=1}^{j-1} X_l \prod_{l=j+1}^{N} Y_l\right] \\ &= \left(1+\varepsilon_j(n,m)\right)\frac{g_{j,N}}{n^{1+\gamma_j}}\mathbf{E}\left[Z_j(m)\mathbf{s}^{\mathbf{Z}(m)} y_j^{Z_j(m)} \prod_{l=1}^{j-1} X_l \prod_{l=j+1}^{N} Y_l\right] \qquad (41) \\ &= \left(1+\tilde{\varepsilon}_j(n,m)\right)\frac{g_{j,N}}{n^{1+\gamma_j}}\mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=1}^{N}\left(\frac{\lambda_l}{L_l(m)} + \frac{c_{l,N}}{(n-m)^{\gamma_l}}\right)Z_l(m)\right\}\right]. \end{aligned}$$

Consider first the case $m \ll n^{\gamma_1}$ and let $L_l(m) = m^l$. Such a choice of parameters reduces (41) to

$$C_j(m,n) = \left(1+\varepsilon_j(n,m)\right)\frac{g_{j,N}}{n^{1+\gamma_j}}\mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=1}^{N}\lambda_l\frac{Z_l(m)}{m^l}\right\}\right]. \qquad (42)$$

These considerations lead to the following statement.

Lemma 18 *If $n^{\gamma_1} \gg m \to \infty$, then, for all $\lambda_l \geq 0, l = 1, ..., N$*

$$\lim_{m\to\infty}\mathbf{E}\left[\exp\left\{-\sum_{l=1}^{N}\lambda_l\frac{Z_l(m)}{m^l}\right\}\Big| T_N = n\right] = \frac{\partial\Phi_1(\lambda_1,\lambda_2,...,\lambda_N)}{\partial\lambda_1}.$$

Proof. We need to show that for $s_l = \exp\left\{-\lambda_l/m^l\right\}, l = 1, ..., N,$

$$\lim_{m\to\infty}\frac{\Psi^{(1,N)}(m,n;\mathbf{s})}{\mathbf{P}\left(T_N = n\right)} = \frac{\partial\Phi_1(\lambda_1,\lambda_2,...,\lambda_N)}{\partial\lambda_1}.$$

It follows from (37)–(42), Theorem 1 and the condition $m \ll n^{\gamma_1}$ that for every $j = 1, 2, ..., N$ it is necessary to investigate the asymptotic behavior of the quantity

$$\frac{g_{j,N}}{g_{1,N}}\frac{n^{\gamma_1}}{n^{\gamma_j}}\mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=1}^{N}\lambda_l\frac{Z_l(m)}{m^l}\right\}\right].$$

According to Lemma 17,

$$\begin{aligned} &\lim_{m\to\infty}\frac{g_{j,N}}{g_{1,N}}\frac{n^{\gamma_1}}{n^{\gamma_j}}\mathbf{E}\left[Z_j(m)\exp\left\{-\sum_{l=1}^{N}\lambda_l\frac{Z_l(m)}{m^l}\right\}\right] \\ &\qquad = \frac{\partial\Phi_1(\lambda_1,\lambda_2,...,\lambda_N)}{\partial\lambda_j}\frac{g_{j,N}}{g_{1,N}}\lim_{m\to\infty} n^{\left(1-2^{j-1}\right)\gamma_1} m^{j-1}. \end{aligned}$$

Since

$$\lim_{m\to\infty} n^{\left(1-2^{j-1}\right)\gamma_1} m^{j-1} = \delta_{1j},$$

it follows that

$$\lim_{m\to\infty} \frac{\Psi^{(1,N)}(m,n;\mathbf{s})}{\mathbf{P}\left(T_N = n\right)} = \sum_{j=1}^{N} \lim_{m\to\infty} \frac{G_j\left(m,n;\mathbf{s}\right)}{\mathbf{P}\left(T_N = n\right)} = \frac{\partial \Phi_1(\lambda_1,\lambda_2,...,\lambda_N)}{\partial \lambda_1}.$$

The lemma is proved.

Lemma 19 *If* $m \sim yn^{\gamma_i}, y > 0$, *for some* $i \in \{1,2,...,N-1\}$, *then, for all* $\lambda_l \geq 0, l = i,...,N$

$$\begin{aligned}
&\lim_{m\to\infty} \mathbf{E}\left[\exp\left\{-\sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\} I_{i-1}(m) \middle| T_N = n\right] \\
&= D_{i-1} \frac{g_{i,N}}{g_{1,N}} \frac{\partial}{\partial \lambda_i} \left(\frac{\Phi_i(\lambda_i' y, \lambda_{i+1}' y^2, \lambda_{i+2} y^3, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}} \\
&\quad + D_{i-1} \frac{g_{i+1,N}}{g_{1,N}} \frac{\partial}{\partial \lambda_{i+1}} \left(\frac{\Phi_i(\lambda_i' y, \lambda_{i+1}' y^2, \lambda_{i+2} y^3, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}},
\end{aligned}$$

where $\lambda_i' = \lambda_i + c_{i,N}, \lambda_{i+1}' = \lambda_{i+1} + c_{i+1,N}$.

Proof. Similarly to the proof of the previous lemma, it is necessary to calculate, for each $j \in \{i, i+1, ..., N\}$ the limit, as $m \to \infty$ of the quantity

$$\begin{aligned}
&\frac{g_{j,N}}{g_{1,N}} \frac{n^{\gamma_1}}{n^{\gamma_j}} \mathbf{E}\left[Z_j(m) \exp\left\{-\sum_{l=i}^{i+1} c_{l,N} \frac{Z_l(m)}{n^{(l-i+1)\gamma_l}} - \sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\} I_{i-1}(m)\right] \\
= \quad & \frac{g_{j,N}}{g_{1,N}} \frac{n^{(j-i+1)\gamma_i}}{n^{\gamma_j}} \frac{n^{\gamma_1}}{n^{(j-i+1)\gamma_i}} \mathbf{E}\left[Z_j(m) \exp\left\{-\sum_{l=i}^{i+1} c_{l,N} \frac{Z_l(m)}{n^{(l-i+1)\gamma_l}} - \sum_{l=i}^{N} \lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\} I_{i-1}(m)\right] \\
= \quad & (1+\varepsilon_j(n,m)) D_{i-1} \frac{g_{j,N}}{g_{1,N}} \frac{n^{(j-i+1)\gamma_i}}{n^{\gamma_j}} \frac{\partial}{\partial \lambda_j} \left(\frac{\Phi_i(\lambda_i' y, \lambda_{i+1}' y^2, \lambda_{i+2} y^3, ..., \lambda_N y^{N-i+1})}{y}\right)^{1/2^{i-1}},
\end{aligned}$$

where we have used Lemma 9 at the last step.

Hence the statement of the lemma follows easily, since

$$\lim_{n\to\infty} \frac{n^{(j-i+1)\gamma_i}}{n^{\gamma_j}} = \begin{cases} 1, & \text{if} \quad j = i, i+1, \\ 0, & \text{if} \quad j \neq i, i+1. \end{cases}$$

Corollary 20 *Under conditions of Lemma 19*

$$\lim_{m\to\infty} \mathbf{P}(Z_1(m) + \cdots + Z_{i-1}(m) > 0 | T_N = n) = 0. \tag{43}$$

Proof. Clearly,

$$\begin{aligned}\mathbf{P}\left(Z_1(m)+\cdots+Z_{i-1}(m)=0|T_N=n\right) &= \mathbf{E}\left[I_{i-1}(m)|T_N=n\right] \\ &\geq \mathbf{E}\left[\exp\left\{-\sum_{l=i}^{N}\lambda_l \frac{Z_l(m)}{n^{(l-i+1)\gamma_i}}\right\} I_{i-1}(m)\middle| T_N=n\right]. \quad (44)\end{aligned}$$

Let $\mathbf{0}^{(N-i-1)}$ be an $N-i-1$–dimensional vector all whose components are zeros. It follows from the definition of Φ_i (see (7)) that

$$\frac{\Phi_i(\lambda_i y, \lambda_{i+1}y^2, \mathbf{0}^{(N-i-1)})}{y} = \frac{\Phi_i^*(\lambda_i y, \lambda_{i+1}y^2)}{y},$$

where (recall (8))

$$\frac{\Phi_i^*(\lambda_i y, \lambda_{i+1}y^2)}{y} = \sqrt{\frac{m_{i,i+1}\lambda_{i+1}}{b_i}} \frac{b_i\lambda_i + \sqrt{b_i m_{i,i+1}\lambda_{i+1}}\tanh\left(y\sqrt{b_i m_{i,i+1}\lambda_{i+1}}\right)}{b_i\lambda_i \tanh\left(y\sqrt{b_i m_{i,i+1}\lambda_{i+1}}\right) + \sqrt{b_i m_{i,i+1}\lambda_{i+1}}}.$$

Rather cumbersome calculations (which we omit), basing on the equalities,

$$c_{i,N} = \sqrt{b_i^{-1} m_{i,i+1} c_{i+1,N}},\ c_{1,N} = D_{i-1}\left(c_{i,N}\right)^{1/2^{i-1}},$$

show that at the point $(\lambda_i, \lambda_{i+1}) = (c_{i,N}, c_{i+1,N})$

$$D_{i-1}\frac{g_{i,N}}{g_{1,N}}\frac{\partial}{\partial\lambda_i}\left(\frac{\Phi_i^*(\lambda_i y, \lambda_{i+1}y^2)}{y}\right)^{1/2^{i-1}} + D_{i-1}\frac{g_{i+1,N}}{g_{1,N}}\frac{\partial}{\partial\lambda_{i+1}}\left(\frac{\Phi_i^*(\lambda_i y, \lambda_{i+1}y^2)}{y}\right)^{1/2^{i-1}} = 1.$$

Combining this result with (44) and Lemma 19 gives

$$\liminf_{m\to\infty} \mathbf{P}\left(Z_1(m)+\cdots+Z_{i-1}(m)=0|T_N=n\right) \geq 1.$$

Thus,

$$\lim_{m\to\infty} \mathbf{P}\left(Z_1(m)+\cdots+Z_{i-1}(m)>0|T_N=n\right) = 0.$$

Corollary is proved.

Lemma 21 *If $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ for some $i \in \{1, 2, ..., N-1\}$, then, for any $\lambda_l \geq 0, l = i+1, ..., N$*

$$\begin{aligned}\lim_{m\to\infty} \mathbf{E}&\left[\exp\left\{-\sum_{l=i+1}^{N}\lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}}\right\} I_i(m)\middle| T_N=n\right] \\ &= \frac{D_i}{2^i}\frac{g_{i+1,N}}{g_{1,N}}\left(c_{i+1,N} + \sum_{l=i+1}^{N}\lambda_l a_{i+1,l}\right)^{-1+1/2^i}.\end{aligned}$$

Proof. Recalling (41) and Lemma 14, we see that it is necessary to calculate for $j \geq i+1$ the limit

$$\lim_{m\to\infty} \frac{g_{j,N}}{g_{1,N}} \frac{n^{\gamma_1}}{n^{\gamma_j}} \mathbf{E}\left[Z_j(m) \exp\left\{ -c_{i+1,N} \frac{Z_{i+1}(m)}{n^{\gamma_{i+1}}} - \sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} I_i(m) \right]$$

$$= \lim_{m\to\infty} \frac{g_{j,N}}{g_{1,N}} \frac{n^{\gamma_{i+1}} m^{j-i-1}}{n^{\gamma_j}} \frac{n^{\gamma_1}}{n^{\gamma_{i+1}} m^{j-i-1}}$$

$$= \times \mathbf{E}\left[Z_j(m) \exp\left\{ -c_{i+1,N} \frac{Z_{i+1}(m)}{n^{\gamma_{i+1}}} - \sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} I_i(m) \right]$$

$$= \frac{D_i a_{i+1,j}}{2^i} \frac{g_{j,N}}{g_{1,N}} \left(c_{i+1,N} + \sum_{l=i+1}^{N} \lambda_l a_{i+1,l} \right)^{-1+1/2^i} \lim_{m\to\infty} \frac{n^{\gamma_{i+1}}}{n^{\gamma_j}} m^{j-i-1}.$$

If $j = i+1$, then

$$\lim_{m\to\infty} \frac{n^{\gamma_{i+1}}}{n^{\gamma_j}} m^{j-i-1} = 1.$$

If $j > i+1$, then

$$\lim_{m\to\infty} \frac{n^{\gamma_{i+1}}}{n^{\gamma_j}} m^{j-i-1} = 0$$

in view of the estimates

$$\frac{n^{\gamma_{i+1}}}{n^{\gamma_j}} m^{j-i-1} \ll \frac{n^{\gamma_{i+1}(j-i)}}{n^{\gamma_j}} = n^{\gamma_{i+1}\left(j-i-2^{j-i-1}\right)} \leq 1.$$

Thus,

$$\lim_{m\to\infty} \mathbf{E}\left[\exp\left\{ -\sum_{l=i+1}^{N} \lambda_l \frac{Z_l(m)}{n^{\gamma_{i+1}} m^{l-i-1}} \right\} I_i(m) \middle| T_N = n \right]$$

$$= \frac{D_i}{2^i} \frac{g_{i+1,N}}{g_{1,N}} \left(c_{i+1,N} + \sum_{l=i+1}^{N} \lambda_l a_{i+1,l} \right)^{-1+1/2^i}.$$

Lemma 21 is proved.

Corollary 22 *Under conditions of Lemma 21*

$$\lim_{m\to\infty} \mathbf{P}(Z_1(m) + \cdots + Z_i(m) > 0 | T_N = n) = 0.$$

Proof. In virtue of Lemma 6 in [16] and the equalities $g_{k,N} = \gamma_k c_{k,N}, \gamma_{i+1} = 2^i \gamma_1$, we have

$$\frac{D_i}{2^i} \left(c_{i+1,N} \right)^{-1+1/2^i} \frac{g_{i+1,N}}{g_{1,N}} = \frac{D_i \left(c_{i+1,N} \right)^{1/2^i} \gamma_{i+1}}{2^i \gamma_1 c_{1,N}} = \frac{D_i \left(c_{i+1,N} \right)^{1/2^i}}{c_{1,N}} = 1,$$

which in view of Lemma 21 finishes the proof.

Lemma 23 *If $m\sim xn, x\in(0,1)$, then, for $\lambda_N\geq 0$*

$$\begin{aligned}&\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[\left(1-\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}\right)I_{N-1}(m)\right]\\&\qquad=\lim_{n\to\infty} n^{\gamma_1}\mathbf{E}\left[1-\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}\right]=\frac{c_{1,N}}{x^{\gamma_1}}\left(1-\frac{1}{1+x\lambda_N}\right)^{\gamma_1}.\end{aligned}$$

Proof. This statement follows from Theorem 4 in [16] and the asymptotic representation (22).

Lemma 24 *If $m\sim xn, x\in(0,1)$, then for $\lambda_N\geq 0$*

$$\begin{aligned}&\lim_{m\to\infty}\mathbf{E}\left[\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}I_{N-1}(m)\Big|\, T_N=n\right]\\&\qquad=\lim_{m\to\infty}\mathbf{E}\left[\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}\Big|\, T_N=n\right]\\&\qquad=\frac{1}{(1+(1-x)\lambda_N)^{1-\gamma_1}}\frac{1}{(1+x(1-x)\lambda_N)^{1+\gamma_1}}.\end{aligned}$$

Proof. Similarly to the proof of (41) one can show, using the notations from (39), that for $m\sim xn, x\in(0,1)$

$$\begin{aligned}&\mathbf{E}\left[\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}I_{N-1}(m);\, T_N=n\right]\\=\;&\mathbf{E}\left[\exp\left\{-\lambda_N\frac{Z_N(m)}{b_N n}\right\}\left(x_N^{Z_N(m)}-y_N^{Z_N(m)}\right)I_{N-1}(m)\right]\\=\;&\frac{(1+\varepsilon_1(m,n))g_{N,N}}{(n(1-x))^2}\mathbf{E}\left[Z_N(m)\exp\left\{-\left(\lambda_N+\frac{b_N c_{N,N}}{1-x}\right)\frac{Z_N(m)}{b_N n}\right\}I_{N-1}(m)\right].\end{aligned}$$

By Lemma 23 for $\lambda\geq 0$ we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{n^{\gamma_1}}{b_N n}\mathbf{E}\left[Z_N(m)\exp\left\{-\lambda\frac{Z_N(m)}{b_N n}\right\}I_{N-1}(m)\right]\\&\quad=\lim_{n\to\infty}\frac{\partial}{\partial\lambda}n^{\gamma_1}\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z_N(m)}{b_N n}\right\}\right)I_{N-1}(m)\right]\\&\quad=\frac{\partial}{\partial\lambda}\lim_{n\to\infty}n^{\gamma_1}\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z_N(m)}{b_N n}\right\}\right)I_{N-1}(m)\right]\\&\quad=\frac{\partial}{\partial\lambda}\frac{c_{1,N}}{x^{\gamma_1}}\left(1-\frac{1}{1+x\lambda}\right)^{\gamma_1}=\gamma_1 c_{1,N}\left(\frac{\lambda}{1+x\lambda}\right)^{\gamma_1-1}\frac{1}{(1+x\lambda)^2}.\end{aligned}$$

Hence, taking into account the equalities $g_{N,N}=b_N^{-1}$, $g_{1,N}=\gamma_1 c_{1,N}$, $b_N c_{1,N}=1$,using the relation

$$\mathbf{P}(T_N=n)\sim\frac{g_{1,N}}{n^{1+\gamma_1}},$$

and setting

$$\lambda=\lambda_N+\frac{b_N c_{N,N}}{1-x}=\lambda_N+\frac{1}{1-x},$$

after evident simplifications we obtain

$$\lim_{n\to\infty}\frac{\gamma_1 c_{1,N} b_N g_{N,N} n^{\gamma_1+1}}{g_{1,N}(n(1-x))^2 b_N}\mathbf{E}\left[Z_N(m)\exp\left\{-\left(\lambda_N+\frac{1}{1-x}\right)\frac{Z_N(m)}{b_N n}\right\} I_{N-1}(m)\right]$$
$$=\frac{1}{\left(1+(1-x)\,\lambda_N\right)^{1-\gamma_1}}\frac{1}{\left(1+x\,(1-x)\,\lambda_N\right)^{1+\gamma_1}},$$

as required.

Corollary 25 *Under conditions of Lemma 24*

$$\lim_{m\to\infty}\mathbf{P}(Z_1(m)+\cdots+Z_{N-1}(m)>0|T_N=n)=0.$$

Proofs of Theorems 2–5. The statements of Theorems 2–5 follow in an evident way from Lemmas 18, 19, 21, 24 and Corollaries 20, 22, 25.

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